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Satellite potentials for hypergeometric Natanzon potentials

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Abstract

As a result of the $so(2,1)$ treatment of the hypergeometric Natanzon potentials V_N a set of potentials related to a given one is determined; these are the satellite potentials and all belong to the Natanzon class. The set arises as the result of the action of the $so(2,1)$ generators on the carrier space of an irreducible representation. The results are compared to those obtained from supersymmetric quantum mechanics (SUSYQM) for some Natanzon potentials. The chains of Natanzon potentials constructed using the idea of a satellite potential are, in most cases, different from the SUSYQM chains.

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1. Introduction

In [1] the $so(2,1)$ description of the hypergeometric Natanzon [2] potentials V_N was presented. The study of the discrete spectrum led to the result that three parameters (called group parameters) are required to completely describe the eigenstates. Two of these parameters correspond to labels of states in a particular irreducible representation (irrep) of $so(2,1)$: the eigenvalues q of the Casimir operator and m of the compact generator. The third parameter, p, labels a particular set of $so(2,1)$ generators. It is found that in most cases there is one eigenstate for each value of p and as a result different sets of $so(2,1)$ generators, each for a given (allowed) value of p, are necessary to describe the eigenfunctions of V_N. It is unusual for all the eigenstates of V_N to belong to a single irrep, so if attention is fixed in a particular irrep, the states connected by the $so(2,1)$ generators are, in the majority of cases, associated with different Natanzon potentials. It is this fact that suggests the definition of a satellite potential. The construction of the set of satellite potentials associated with a given V_N is the subject of this paper. The term satellite potential is used as in [3].

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The action of the $so(2, 1)$ generators defines a set of potentials which share the property that the eigenstates connected by them have the same p and q but correspond, in general, to different energy eigenvalues. If the energy eigenvalues of the states connected by the $so(2, 1)$ generators are the same, the following question naturally arises: are the $so(2, 1)$ generators related to the operators in supersymmetric quantum mechanics (SUSYQM)? This is not so for all the shape-invariant potentials V_S [5], as is shown below. It is also proven that the state reached by the action of the $so(2, 1)$ generators is a Natanzon potential state. This implies that the Natanzon class of potentials is invariant under this algebra, a result different to the one obtained in SUSYQM due to the fact that the SUSYQM operators may define a potential that does not belong to the Natanzon class [8]; this happens for the Ginocchio potential. It is proven below that the action of the $so(2, 1)$ generators on an eigenstate of a Ginocchio potential defines a Ginocchio potential only for particular choices of the Natanzon parameters of the satellite. The potential obtained is still in the Natanzon class and this is obviously different from the result obtained from SUSYQM.

Since the action of the $so(2, 1)$ generators does not define a potential outside the Natanzon class, the most general result that can be expected from the definition of satellite potentials is the determination of a new sequence of Natanzon potentials starting from a given one. In the majority of cases studied, the set of satellite potentials includes potentials that belong to a subclass of the original one; this happens for the Rosen–Morse (RM), Pöschl–Teller and Eckart potentials [6]; the Ginocchio potential may be an exception.

It has to be stressed that $so(2, 1)$ is the underlying algebra in the description of the Natanzon class of potentials; no other algebraic scheme is necessary to completely describe the energy spectrum and wavefunctions of this class of potentials. Therefore, the most general consequence of this algebraic description is that the satellite potentials belong to a, generally, new sequence of potentials as compared to the one defined by SUSYQM and that all potentials belong to the Natanzon class. This being so, the aim is to describe as precisely as possible the sequence obtained by the action of the $so(2, 1)$ generators.

The paper is organized as follows: section 2 gives a brief description of the bound state sector for the Natanzon potential; in section 3 the satellite potentials are introduced; and section 4 includes several examples.

2. The $so(2, 1)$ description of the hypergeometric Natanzon potentials

A two-variable realization of the algebra $so(2, 1)$ is used (for details see [1]) with the generators taken as

$$\begin{aligned} e^{\pm i\phi} J_{\mp} &= -\frac{i(1+z)}{2\sqrt{z}}\frac{\partial}{\partial\phi} \mp \frac{\sqrt{z}(z-1)}{z'}\frac{\partial}{\partial r} + \left(\mp\frac{\sqrt{z}(z-1)z''}{2z'^2} + \frac{(z-1)(\pm 1+p)}{2\sqrt{z}}\right) \\ J_0 &= -i\frac{\partial}{\partial\phi} \end{aligned} \tag{1}$$

where $z = z(r)$ and $z' = dz/dr$. Expressions (1) lead to the Casimir, $Q = J_0(J_0 + 1) - J_- J_+$:

$$Q = (z-1)^2\left[\frac{z}{z'^2}\frac{\partial^2}{\partial r^2} + \frac{1}{4z}\frac{\partial^2}{\partial\phi^2} + \frac{ip(1+z)}{2z(z-1)}\frac{\partial}{\partial\phi} + \frac{-p^2+1}{4z} + \frac{zz'''}{2z'^3} - 3\frac{zz''^2}{4z'^4}\right]. \tag{2}$$

The significant result that follows from (1), (2) is the appearance of the parameter p in the explicit expression for the generators and the Casimir; this constant distinguishes a particular set of $so(2, 1)$ generators and plays a crucial role in the $so(2, 1)$ description of the Natanzon potentials.

The physical problem dealt with is the derivation of the discrete spectrum of the Hamiltonian and to this end the compact operator J_0 is diagonalized; the irreps of $so(2, 1)$

considered are unitary and therefore infinite dimensional, and of these the relevant one is the one bounded below (the D^+ representation). In this representation the eigenvalue m of the compact operator is given in terms of the eigenvalue q of the Casimir and the counter $\nu = 0, 1, \ldots$ as

$$m(\nu) = \nu + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{4q+1}. \tag{3}$$

With the above results, the $so(2,1)$ description of the Schrödinger equation is defined by (4), referred to as the master equation:

$$G(r)(Q-q)\Psi(r,\phi) = (E-H)\Psi(r,\phi) \tag{4}$$

where H is the Hamiltonian and q, E are the eigenvalues of the Casimir and Hamiltonian, respectively. The function $G(r)$ ensures that the coefficients of the second derivatives of $\Psi(r,\phi)$ are the same on both sides. From (4) it follows that, in general, q could be a function of ν which also labels E; in spite of the fact that p does not appear explicitly in (4), it could also depend on ν. Each eigenfunction of the Casimir (or equivalently of the Hamiltonian) has the form

$$\Psi(r,\phi) = \exp(\mathrm{i}m(\nu)\phi)g \tag{5}$$

and is also an eigenfunction of the compact operator J_0. The function $g = g(r)$ is determined by solving the master equation.

The set of Natanzon hypergeometric potentials [2] is given by

$$V_N = \frac{fz^2 - (h_0 - h_1 + f)z + h_0 + 1}{R} + \left[a + \frac{a + (c_1 - c_0)(2z-1)}{z(z-1)} - \frac{5\Delta}{4R}\right]\left[\frac{z(1-z)}{R}\right]^2 \tag{6}$$

where $(a, c_0, c_1, f, h_0, h_1)$ are the Natanzon parameters and z is a solution of the differential equation

$$z(r)' = 2\frac{z(1-z)}{\sqrt{R}}; \tag{7}$$

the other symbols that appear in (6), (7) are given by

$$\tau = c_1 - c_0 - a, \qquad \Delta = \tau^2 - 4ac_0, \qquad R = az^2 + \tau z + c_0. \tag{8}$$

The $so(2,1)$ description of $V_N(r)$ is obtained after the explicit expression for the Casimir (2) is put into a form similar to (6) after use of (7) and the coefficients of the powers of z compared; this leads to

$$\begin{aligned} p(\nu) + m(\nu) &= \sqrt{-aE(\nu) + f + 1} = \alpha(\nu), \\ p(\nu) - m(\nu) &= \sqrt{-c_0 E(\nu) + h_0 + 1} = \beta(\nu), \\ \sqrt{4q(\nu)+1} &= \sqrt{-c_1 E(\nu) + h_1 + 1} = \delta(\nu) \end{aligned} \tag{9}$$

where $\alpha(\nu)$ is shorthand for $\sqrt{-aE(\nu) + f + 1}$ and similarly for $\beta(\nu)$ and $\delta(\nu)$. From (3), (9), $p(\nu)$, $q(\nu)$, $m(\nu)$—called the group parameters—and $E(\nu)$ are determined for each value of ν, thus fixing a particular position in an $so(2,1)$ irrep and the energy eigenvalue. This implies that one eigenfunction of a specific V_N (characterized by a particular set of Natanzon parameters) belongs to the carrier space of an $so(2,1)$ labelled by $p(\nu)$ (see (2)) and in this carrier space its position is specified by $q(\nu)$ and $m(\nu)$. From now on $p(\nu)$, $q(\nu)$ and $m(\nu)$ are written as p, q and m. For any given irrep, it is possible that its states are eigenfunctions of different Natanzon potentials. The characterization of this sequence of Natanzon potentials is the main purpose of the present study. The carrier space of a specific irrep is given by

$$\Psi_{pqm}(r,\phi) = \exp(\mathrm{i}m\phi)\Phi_{pqm}(r) \tag{10}$$

where

$$\Phi_{pqm}(r) = Kz^{\beta(\nu)/2}\,(1-z)^{\delta(\nu)/2}\,R^{1/4}{}_2F_1(-\nu, \alpha(\nu)-\nu; 1+\beta(\nu); z). \tag{11}$$

with K a normalization constant. The energy spectrum is obtained, using (3) and (9), from

$$\alpha(\nu) - \beta(\nu) - \delta(\nu) = 2\nu + 1. \tag{12}$$

3. Satellite potentials

In this section the possibilities allowed by (9) for fixed values of the group parameters (p, q) are presented. The study of this situation leads to the construction of the satellite potentials. The action of the $so(2, 1)$ generators on (11) is the following:

$$\begin{aligned} J_-\Psi_{pqm} &= \frac{\nu(\alpha(\nu)-\nu-1-\beta(\nu))}{1+\beta(\nu)}\Psi_{pqm-1} \\ J_+\Psi_{pqm} &= -\beta(\nu)\Psi_{pqm+1} \end{aligned} \tag{13}$$

which follow from [7]

$$\begin{aligned} {}_2F_1(a+1, b+1, c+1; z) = -\frac{1}{zb(z-1)a}[&c(c-1){}_2F_1(a-1, b, c-1; z) \\ &- c(zb-c+1){}_2F_1(a, b, c; z)] \end{aligned} \tag{14}$$

and

$$\begin{aligned} &{}_2F_1(a-1, b, c-1; z) \\ &\quad = -\frac{1}{(c-1)}[(b-c+1){}_2F_1(a, b, c; z) - b(z-1){}_2F_1(a, b+1, c; z)]. \end{aligned} \tag{15}$$

It follows from (13) that the action of the $so(2, 1)$ generators on an eigenfunction of a given Natanzon potential gives an eigenfunction of a different Natanzon potential. In fact, the same z is present, so a, c_0, c_1 are unchanged. Since p and q are unchanged and $m \to m \pm 1$ (9) leads to

$$\alpha_S(\nu \pm 1) = \alpha(\nu) \pm 1, \qquad \beta_S(\nu \pm 1) = \beta(\nu) \mp 1, \qquad \delta_S(\nu \pm 1) = \delta(\nu) \tag{16}$$

where $\alpha_S(\nu \pm 1)$, $\beta_S(\nu \pm 1)$, $\delta_S(\nu \pm 1)$ are obtained from (9) by the replacement $f \to f_S$, $h_0 \to h_{0S}$, $h_1 \to h_{1S}$, $E(\nu) \to E_S(\nu \pm 1)$, which are the Natanzon parameters and energy for the satellite potential, and $\nu \to \nu \pm 1$. These relations may be used to determine, for example, the new parameters f_S, h_{0S} and $E_S(\nu \pm 1)$ as functions of ν. The freedom in the choice of one of the parameters is a consequence of having three equations, (16), with four unknowns; we choose h_{1S} as the free parameter. This is a general situation found in the study of the Natanzon potentials where usually one parameter is used to fix the asymptotic behaviour at infinity or the ground state energy of the system.

The potentials to which the eigenfunctions with $m \pm 1$ are associated will be called satellite to the one related to the eigenfunction labelled by m. The common feature of all these potentials is that the values of both p and q remain unchanged. It is natural to ask how many satellite potentials are associated with a given one? Different eigenfunctions of a given Natanzon potential do not necessarily belong to the same $so(2, 1)$ irrep and therefore for each such function a certain number of satellite potentials are determined. We fix our attention on one eigenfunction. It belongs at the same time to the set of eigenfunctions of V_N and to the $so(2, 1)$ irrep. Recall that ν labels the position of $\Phi_{pqm}(r)$ in the irrep and call λ the label of its position in the set of eigenfunctions of V_N. The above description assigns a set of values of (p, q, m) in each case and the two sets must coincide; that is,

$$p(\nu) = p(\lambda), \qquad q(\nu) = q(\lambda), \qquad m(\nu) = m(\lambda) \tag{17}$$

which imply

$$\lambda = \nu \tag{18}$$

so the numerical values of the two labels coincide on the irrep. Therefore, the action of J_+ increases both ν and λ and if there is a maximum value of λ for a given potential a finite number of satellite potentials will be constructed. The maximum value λ depends on whether the $r \to \infty$ limit is finite or not for each of the satellite potentials.

To complete the answer, consider a Natanzon potential V_N one of whose eigenfunctions, $\Psi^{\nu_0}_{\lambda_0=\nu_0}$, is in the (p, q) irrep; its position in the set of eigenfunctions is $\lambda = \lambda_0$ and the one in the irrep is $\nu = \nu_0$. The reason for introducing both ν and λ (which may seem redundant because for this particular function their numerical values coincide) is that, as shown above, the Natanzon parameters of the satellite to V_N depend on those for V_N in a way determined by (16); therefore, these parameters may (and in fact will) be functions of ν. Thus, displacement along the irrep produces sets of Natanzon parameters that change with ν in such a way that their numerical values coincide with those for V_N when $\nu = \nu_0$ and those for the satellite when $\nu = \nu_0 \pm 1$. Moving from one eigenfunction of V_N to another keeps ν_0 fixed since in this case the Natanzon parameters do not change. The question now is the following: if $\Psi^{\nu_0}_{\lambda_0=\nu_0}$ and $\Psi^{\nu_0+1}_{\lambda_0+1=\nu_0+1}$ are in the same irrep, does the same occur for $\Psi^{\nu_0}_{\lambda_0+1}$ and $\Psi^{\nu_0+1}_{\lambda_0+2}$? To answer this question the system (9), (3) is studied, taking the Natanzon parameters for $\nu = \nu_0$ and $\nu = \nu_0 + 1$, replacing ν by λ and comparing the values of p and q obtained for $\lambda = \lambda_0 + 1$ (for ν_0) and $\lambda = \lambda_0 + 2$ (for $\nu_0 + 1$). It turns out that in general these values are not the same; the analysis described above has to be repeated for each eigenfunction and a new set of satellite potentials is thus determined.

4. Particular cases

The results that follow include the set of Natanzon parameters for the potential and its supersymmetric partner and the rule that generates the parameters for the satellite potentials. A detailed study along the lines presented in this paper has been performed for the Eckart potential in [6] where it is shown that the satellite potential does not coincide with the supersymmetric partner.

(1) The Pöschl–Teller II potential in the notation of [9] is

$$V_{PT2} = (A - B)^2 - A(A + \alpha)\,\mathrm{sech}(\alpha r)^2 + B(B - \alpha)\,\mathrm{cosech}(\alpha r)^2. \tag{19}$$

The Natanzon parameters

$$\begin{aligned} &a = 0, \qquad c_0 = 0, \qquad c_1 = \alpha^{-2}, \\ &f = \frac{(2A - \alpha)(2A + 3\alpha)}{4\alpha^2}, \\ &h_0 = \frac{(2B + \alpha)(2B - 3\alpha)}{4\alpha^2}, \\ &h_1 = \frac{(A - B + \alpha)(A - B - \alpha)}{\alpha^2} \end{aligned} \tag{20}$$

reproduce V_{PT2} with $z = \tanh(\alpha r)^2$ replaced in (6). From (9), (12) and (20) the energy spectrum is

$$E(\nu) = -4\alpha\nu(\nu\alpha - A + B) \tag{21}$$

while the group parameters are from (9)

$$p(\nu) = \frac{A + B}{2\alpha}, \qquad q(\nu) = \frac{(A - B - 2\nu\alpha)^2 - \alpha^2}{4\alpha^2}, \qquad m(\nu) = \frac{A - B + \alpha}{2\alpha} \tag{22}$$

and from the definition of $\alpha(\nu)$, $\beta(\nu)$ and $\delta(\nu)$,

$$\alpha(\nu) = \frac{\alpha + 2A}{2\alpha}, \qquad \beta(\nu) = \frac{2B - \alpha}{2\alpha}, \qquad \delta(\nu) = -\frac{2\nu\alpha - A + B}{\alpha}. \tag{23}$$

Remark. If A and B are kept unchanged, ν is replaced by λ to label the position of the eigenfunction for a given V_{PT2} and $\lambda \to \lambda \pm 1$, it follows from (22) that $p(\lambda \pm 1) = p(\lambda)$ and $m(\lambda \pm 1) = m(\lambda)$ while $4\alpha^2 q(\lambda \pm 1) = (A - B - 2(\lambda \pm 1)\alpha)^2 - \alpha^2$. The eigenfunction labelled by $\lambda \pm 1$ belongs to a different irrep to the one labelled by λ.

After use of (16), the action of J_+ leads to

$$\begin{aligned} \alpha_S(\nu + 1) &= \frac{3\alpha + 2A}{2\alpha}, \qquad \beta_S(\nu + 1) = \frac{2B - 3\alpha}{2\alpha}, \\ \delta_S(\nu + 1) &= -\frac{2\nu\alpha - A + B}{\alpha}. \end{aligned} \tag{24}$$

Calling (A_S, B_S) the parameters of the potential in (23) and equating the result with the values in (24), we obtain

$$A_S = A + \alpha, \qquad B_S = B - \alpha. \tag{25}$$

This is similar to the change of parameters when constructing the supersymmetric partner [4,9]. The energy spectrum for the satellite potential follows from (12):

$$E_S(\nu + 1) = E(\nu) + \alpha^2(1 + h_{1S}) - (A - B)^2 \tag{26}$$

with h_{1S} arbitrary; if the condition that the energy vanishes for $\nu = 0$ is imposed, then

$$h_{1S} = \frac{(-A + B - \alpha)(\alpha + B - A)}{\alpha^2} \tag{27}$$

which leads to $E_S(\nu + 1) = E(\nu)$. The result for $E_S(\nu + 1)$ is not the one encountered in SUSYQM because the arguments appear in reverse order; this suggests using J_- instead of J_+. To check if this is appropriate, the supersymmetric partner is now considered.

The supersymmetric partner of V_{PT2}, V_{PPT2}, is obtained from the superpotential [4] $W(r) = A\tanh(\alpha r) - B\coth(\alpha r)$ using $V_{PPT2} = W(r)^2 + W(r)'$, where $W(r)' = \mathrm{d}W(r)/\mathrm{d}r$, with the result

$$V_{PPT2} = (A - B)^2 + A(-A + \alpha)\,\mathrm{sech}(\alpha r)^2 + B(B + \alpha)\,\mathrm{cosech}(\alpha r)^2. \tag{28}$$

The Natanzon parameters for V_{PPT2} are (the subindex p refers to the SUSYQM partner)

$$\begin{aligned} &a = 0, \qquad c_0 = 0, \qquad c_1 = \alpha^{-2}, \\ &f_p = \frac{(2A - 3\alpha)(2A + \alpha)}{4\alpha^2} \\ &h_{p0} = \frac{(2B - \alpha)(2B + 3\alpha)}{4\alpha^2} \\ &h_{p1} = \frac{(A - B + \alpha)(A - B - \alpha)}{\alpha^2} \end{aligned} \tag{29}$$

with the same z as before. The energy spectrum is $E_p(\nu) = -4\alpha(\nu + 1)(\alpha(\nu + 1) - A + B)$. From (21) there follows the standard result for supersymmetric partners: $E_p(\nu) = E(\nu + 1)$.

Repeating the calculation of the group parameters, we find for the supersymmetric partner

$$\begin{aligned} p_p(\nu) &= \frac{A + B}{2\alpha}, \qquad m_p(\nu) = \frac{A - B - \alpha}{2\alpha}, \\ q_p(\nu) &= \frac{(A - B - 2\nu\alpha - 3\alpha)(A - B - 2\nu\alpha - \alpha)}{4\alpha^2}. \end{aligned} \tag{30}$$

Also from (9),

$$\alpha_p(\nu) = \frac{2A-\alpha}{2\alpha}, \qquad \beta_p(\nu) = \frac{2B+\alpha}{2\alpha}, \qquad \delta_p(\nu) = -\frac{2\alpha(\nu+1)-A+B}{\alpha}. \tag{31}$$

Comparing (31) with (22) and (23), the relations between the group parameters of the potential (19) and its SUSYQM partner are as follows:

$$\begin{aligned} p_p(\nu) &= p(\nu), & m_p(\nu) &= m(\nu)-1, \\ q_p(\nu) &= q(\nu+1), & \delta_p(\nu) &= \delta(\nu+1). \end{aligned} \tag{32}$$

The results given in (31) should be compared with the basic conditions on the parameters of the satellite potentials, equations (16) with the lower sign. This implies that $\delta_s(\nu-1) = \delta(\nu)$; then from (32) it is seen that $\delta_s(\nu-1) = \delta_p(\nu-1) = \delta(\nu)$. Using the last equation in (9), it follows that $q_S(\nu-1) = q_p(\nu-1) = q(\nu)$. The result for $p_S(\nu-1)$ for the satellite is, from (16) and (23), $p_S(\nu-1) = p(\nu)$. Also $m_p(\nu) = m_S(\nu) = m(\nu)$. This shows that the potential reached by the action of J_- is the SUSYQM partner. It has to be noticed that this occurs in this particular case and is not valid in general as the following cases show.

(2) The RM potential

$$V_{RM} = A^2 + \frac{B^2}{A^2} + 2B\tanh(\alpha r) - A(A+\alpha)\,\mathrm{sech}(\alpha r)^2 \tag{33}$$

is obtained from (6) with the Natanzon parameters

$$\begin{aligned} &a = 0, \qquad c_0 = c_1 = 1/\alpha^2, \\ &f = 4\frac{A(A+\alpha)}{\alpha^2}, \end{aligned} \tag{34}$$

$$\begin{aligned} h_0 &= \frac{(-B+A\alpha+A^2)(-B-A\alpha+A^2)}{\alpha^2A^2}, \\ h_1 &= \frac{(B+A\alpha+A^2)(B-A\alpha+A^2)}{\alpha^2A^2} \end{aligned} \tag{35}$$

with $z = 1/2 + \tanh(\alpha r)/2$. From (12) the energy spectrum is given by

$$E(\nu) = A^2 - (A-\nu\alpha)^2 + \frac{B^2}{A^2} - \frac{B^2}{(A-\nu\alpha)^2}. \tag{36}$$

The group parameters are found to be

$$\begin{aligned} p(\nu) &= \frac{2A}{\alpha} - m(\nu) + 1, \\ q(\nu) &= (\nu+1-m(\nu))(\nu-m(\nu)), \\ m(\nu) &= \frac{-A^2+\nu\alpha^2-A\alpha-B+\nu^2\alpha^2}{2(\nu\alpha-A)\alpha}. \end{aligned} \tag{37}$$

Remark. Keeping A and B fixed and replacing ν by λ, it follows that $p(\lambda\pm1) \neq p(\lambda)$, $q(\lambda\pm1) \neq q(\lambda)$ and $m(\lambda\pm1) \neq m(\lambda)$.

From (9), (37) it follows that

$$\begin{aligned} \alpha(\nu) &= \frac{2A+\alpha}{\alpha}, \\ \beta(\nu) &= \frac{\nu^2\alpha^2-2\nu\alpha A+A^2-B}{(A-\nu\alpha)\alpha}, \\ \delta(\nu) &= \frac{A^2+B+\nu^2\alpha^2-2\nu\alpha A}{(A-\nu\alpha)\alpha}. \end{aligned} \tag{38}$$

For the satellite potential it is found from (16) that

$$\begin{aligned}
\alpha_S(\nu+1) &= 2\frac{A+\alpha}{\alpha}, \\
\beta_S(\nu+1) &= \frac{A-\nu\alpha}{\alpha} - 1 - \frac{B}{(A-\nu\alpha)\alpha}, \\
\delta_S(\nu+1) &= \frac{A-\nu\alpha}{\alpha} + \frac{B}{(A-\nu\alpha)\alpha}.
\end{aligned} \tag{39}$$

Setting (A_S, B_S, p_S, q_S) instead of (A, B, p, q) in (37) with $\nu \to \nu + 1$ and requiring that $p(\nu) = p_S(\nu+1)$, $q(\nu) = q_S(\nu+1)$, it is found that

$$A_S = A + \alpha/2, \qquad B_S = -\frac{(2\nu\alpha + \alpha - 2A)(\nu\alpha^2 - A\alpha - 2B)}{4(\nu\alpha - A)}. \tag{40}$$

The result in (40) shows that the change in (A, B) is different from the one found for the supersymmetric partner, since in that case only A changes while B remains the same [4, 9]. The energy of the satellite is found to be

$$E_S(\nu+1) = E(\nu) + (h_{1S} - h_1)\alpha^2 \tag{41}$$

with h_{1S} arbitrary.

(3) The Ginocchio potential has Natanzon parameters [4]

$$\begin{aligned}
&c_0 = 0, && c_1 = \lambda^{-4}, && a = \lambda^{-4} - \lambda^{-2}, \\
&h_0 = -\tfrac{3}{4}, && h_1 = -1, && f = (\mu + 1/2)^2 - 1
\end{aligned} \tag{42}$$

and its explicit expression is, after using (6),

$$V_G = -\frac{(2\mu+1)^2\lambda^4}{4(\lambda-1)(\lambda+1)} - \frac{(-\lambda^2 - 2 + 4\mu^2 + 4\mu)\lambda^4}{4(-1+\lambda^2)L} + \frac{(6\lambda^2+3)\lambda^4}{4(-1+\lambda^2)L^2} + \frac{5\lambda^6}{4(-1+\lambda^2)L^3} \tag{43}$$

with $L = z(-1+\lambda^2) - \lambda^2$ and z the solution of (7), whose expression in terms of r is implicit. The energy spectrum obtained from (12) is

$$E(\nu) = -\tfrac{1}{4}\left(2\nu + 1 - \sqrt{4\lambda^2[\mu(\mu+1) - \nu(\nu+1)] + (2\nu+1)^2}\right)^2 \tag{44}$$

and this coincides with the spectrum in [4]. The group parameters obtained from (9) are

$$\begin{aligned}
&q(\nu) = -\frac{\lambda^4 + E(\nu)}{4\lambda^4}, \qquad p(\nu) = \frac{\mu^2 + 4\nu^2 + \mu}{2(4\nu+1)} + \frac{E(\nu)}{2\lambda^2(4\nu+1)}, \\
&m(\nu) = \nu + \frac{1}{2} + \frac{1}{2\lambda^2}\sqrt{-E(\nu)}.
\end{aligned} \tag{45}$$

Recall that $\alpha(\nu) = p(\nu) + m(\nu)$, $\beta(\nu) = p(\nu) - m(\nu)$ and $\delta(\nu) = \sqrt{4q(\nu)+1}$, and can easily be computed from (45).

Remark. As in the previous examples, it is found that for fixed Natanzon parameters and calling σ the label of one eigenfunction, $p(\sigma \pm 1) \neq p(\sigma)$, $q(\sigma \pm 1) \neq q(\sigma)$ and $m(\sigma \pm 1) \neq m(\sigma)$. Now the satellites are constructed. The Natanzon parameters and energy spectrum are obtained from (16); the result is

$$\begin{aligned}
&-4(\lambda^2 - 1)h_{1S} = 4(\lambda^2 + f_S - \mu^2 - \mu) - 5 - \frac{4}{\lambda^2}\sqrt{\lambda^4(2\mu+1)^2 - 4E(\nu)(1-\lambda^2)}, \\
&h_{0S} = -\tfrac{3}{4}, \\
&4(\lambda^2 - 1)E_S(\nu) = 4\lambda^2\sqrt{4E(\nu)(\lambda^2 - 1) + \lambda^4(2\mu+1)^2} \\
&\qquad + 4E(\nu)(\lambda^2 - 1) + \lambda^4[(2\mu+1)^2 - 4f_S]
\end{aligned} \tag{46}$$

where in this case f_S is left arbitrary. The satellite does not belong to the Ginocchio class if $h_{1S} \neq -1$. If, however, the free parameter is chosen such that $h_{1S} = -1$, then the satellite is also a Ginocchio potential; this is the only case considered. Then

$$f_S = \frac{(2\mu + 1)^2}{4} + \frac{\sqrt{\lambda^4(2\mu + 1)^2 - 4E(\nu)(1 - \lambda^2)}}{\lambda^2} \tag{47}$$

which can be cast in the form (42) $f_S = (\Gamma + 1/2)^2 - 1$ where Γ is the parameter that replaces μ in the satellite. The choice $h_{1S} = -1$ makes the Ginocchio class invariant under $so(2, 1)$ and therefore an invariant subclass of the Natanzon potentials. The eigenstates connected by the generators have the same energy eigenvalue, $E_S(\nu) = E(\nu)$, as seen by replacing (47) in the expression for the energy in (46). The results obtained are obviously different from the ones from SUSYQM where the actions of the SUSY operators define a series of potentials that, starting from a Ginocchio potential, do not belong to the Natanzon class [8].

The main conclusion that follows from the above examples is that the chain of potentials defined by the action of the $so(2, 1)$ generators could be different from the one defined by the action of the SUSYQM operators and of the potential algebra. It is worth stressing that the only algebraic structure underlying the study of satellite potentials is the $so(2, 1)$ algebra and that this algebra does not define a satellite potential that lies outside the Natanzon class; its most important aspect is the possibility of defining new chains of Natanzon potentials.

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